

RESTRICTING CELL MODULES OF PARTITION ALGEBRAS

INGA PAUL

ABSTRACT. The restriction of a Specht module to a smaller symmetric group has a filtration by Specht modules of this smaller group. In the cellular structure of the group algebra of the symmetric group, the cell modules are exactly the Specht modules. The partition algebra is a cellular algebra containing the group algebra of the symmetric group. In this article, we study the structure of the restriction of a cell module to the group algebra of a symmetric group (with smaller index) and show that it has Specht filtration if the characteristic of the field is large enough.

1. INTRODUCTION

The partition algebra was independently defined by Martin [Mar] and Jones [Jon] to describe the Potts model in statistical mechanics. In representation theory, the partition algebra $P_k(r, \delta)$ arises as a diagram algebra containing the Temperley-Lieb and Brauer algebras. It has nice structural properties such as being cellular [Xi] and quasi-hereditary if and only if $\delta \neq 0$ and $\text{char } k = 0$ or $\text{char } k > r$ [KX]. For $r \geq n$, there is Schur-Weyl duality between $k\Sigma_n$ and $P_k(r, n)$, see for example [HR].

An enhanced cellular structure, called *cellularly stratified*, ensures that the cell modules of $P_k(r, \delta)$ with $\delta \neq 0$, arise from cell modules of the group algebras of symmetric groups with index $n \leq r$ by induction, i.e. they are induced (dual) Specht modules. The same holds for the cell modules of the Brauer algebra. When restricted to a group algebra of a symmetric group, the cell modules of a Brauer algebra admit a filtration by (dual) Specht modules, as shown in [Pag]. The approach used for Brauer algebras is not applicable for the partition algebra, since it is based on the fact that, in a Brauer diagram, each dot is connected to exactly one other dot. For the partition algebra, there is no such regularity, which makes the problem more complex.

In this article, we consider the question of when the restriction of a cell module of the partition algebra $A = P_k(r, \delta)$ to a group algebra of a symmetric group Σ_l with $l \leq r$ admits a cell filtration. The main result is the following

Theorem 1. *Let $A = P_k(r, \delta)$ and $n \leq l \leq r$. Let $\text{char } k > \lfloor \frac{r-n}{3} \rfloor$ or $\text{char } k = 0$ and $X \in k\Sigma_n\text{-mod}$. Then the $k\Sigma_l$ -module $e_l(A/Ae_{n-1}A)e_n \otimes_{e_n A e_n} X$ admits a dual Specht filtration. In particular, restrictions of cell modules of A to $k\Sigma_l\text{-mod}$ have dual Specht filtrations.*

Date: June 23, 2016.

2010 Mathematics Subject Classification. 16G10, 20C30 (primary) and 05E10, 20B30, 20G05, 81R05 (secondary).

Key words and phrases. partition algebras, cellular algebras, restriction, Specht filtration, tensor induction, Foulkes module.

where the e_i are special idempotents in the cellularly stratified structure of A . They are defined in Subsection 2.2.

The article is organised as follows. Section 2 contains a definition of the partition algebra, some notation and an explanation of the cellular and cellularly stratified structures. Sections 3 and 4 are dedicated to the proof of Theorem 1. Section 3 is a study of the $(k\Sigma_l, k\Sigma_n)$ -bimodule $e_l A e_n / e_l J_{n-1} e_n = e_l (A / J_{n-1}) e_n$, the left part of the tensor product from Theorem 1, where $J_{n-1} = A e_{n-1} A$. We want to show that $e_l (A / J_{n-1}) e_n$ has a dual Specht filtration as left $k\Sigma_l$ -module. A decomposition into direct summands is given in Subsection 3.1. Each summand is defined via a generating diagram. All these generators have the same bottom row and non-crossing propagating lines, i.e. they are defined by their respective top rows. In Subsection 3.2, we reformulate the summands as a tensor product

$$k\Sigma_l \otimes_{k\Pi_\alpha \times k\Pi_\beta} k\Sigma_n,$$

where we tensor over the stabilizers of certain dots of the top row of the generator from the previous subsection, namely labelled and unlabelled dots. Since this tensor product is not well understood, we state further isomorphisms to end up with a module that is known to have a dual Specht filtration. Step one is to separate the labelled from the unlabelled dots. This gives an induced outer tensor product:

$$k\Sigma_l \otimes_{k\Sigma(l_1, l_2)} ((k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \otimes_{k\Pi_\beta} k)).$$

Step two is to study the modules $k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n$ and $k\Sigma_{l_2} \otimes_{k\Pi_\beta} k$ separately. In Subsection 3.3, we show that $k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n$ is isomorphic to the bimodule $\oplus (k\Sigma_{l_1} \otimes_{k\Sigma_\gamma} k)$, where we define a right $k\Sigma_n$ -module structure such that it is a *tensor induced module*. As left $k\Sigma_l$ -module, this is obviously dual Specht filtered, as it is a direct sum of permutation modules. In Subsection 3.4, we show that $k\Sigma_{l_2} \otimes_{k\Pi_\beta} k$ is isomorphic to an induced outer tensor product of *Foulkes modules* $k\Sigma_{am} \otimes_{k(\Sigma_a \wr \Sigma_m)} k$. If $\text{char } k > m$ (or $\text{char } k = 0$), we can show that the Foulkes module $k\Sigma_{am} \otimes_{k(\Sigma_a \wr \Sigma_m)} k$ has a dual Specht filtration. This leads to the assumption $\text{char } k > \lfloor \frac{r-n}{3} \rfloor$ (or $\text{char } k = 0$) in Theorem 1. Note that this assumption is sufficient, but potentially not necessary. In fact, Giannelli shows in [Gia, Theorem 1.1] that for $0 < \text{char } k \leq m$ there is a non-projective summand of $k\Sigma_{am} \otimes_{k(\Sigma_a \wr \Sigma_m)} k$ which is not a Young module. However, this does not mean that there is no dual Specht filtration of the Foulkes module.

Finally, Section 4 concludes the proof of the theorem by putting together the results from the previous section, the result on Brauer algebras from [Pag] and the characteristic-free version of the Littlewood-Richardson rule [JP].

This article is the first of two articles arising from the author's PhD thesis [Pau]. The aim of the thesis was to extend the construction of $k\Sigma_r$ -permutation modules to permutation modules for a large class of diagram algebras, as it was done by Hartmann and Paget for Brauer algebras [HP]. A list of assumptions which the algebra A has to satisfy was given, including: the cell modules of A admit a dual Specht filtration when restricted to $k\Sigma_l$ -mod. The remaining assumptions are comparatively easy to show for $A = P_k(r, \delta)$, which makes the article at hand the crucial ingredient for the definition of permutation modules for partition algebras.

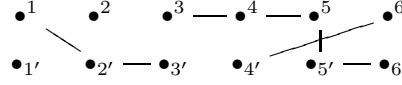
2. DEFINITION OF THE PARTITION ALGEBRA AND ITS STRUCTURE

2.1. Definition. Let k be an algebraically closed field of arbitrary characteristic and let $\delta \in k$. Let $r \in \mathbb{N}$.

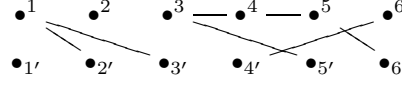
The *partition algebra* $P_k(r, \delta)$ is the associative k -algebra with basis consisting of set partitions of $\{1, \dots, r, 1', \dots, r'\}$. A *set partition* of a set X is a collection of pairwise disjoint subsets $X_i \subseteq X$, such that $\coprod X_i = X$. Regarding $P_k(r, \delta)$ as a diagram algebra, this means that the basis consists of diagrams with two rows of r dots each (top row labelled by $1, \dots, r$ and bottom row labelled by $1', \dots, r'$), where dots which belong to the same part of the partition are connected transitively. Note that this description is not unique. For example, the set partition

$$\{\{1, 2', 3'\}, \{2\}, \{3, 4, 5, 5', 6'\}, \{6, 4'\}, \{1'\}\}$$

corresponds, among others, to the diagram

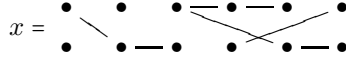


as well as to the diagram

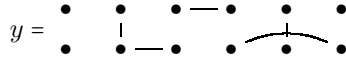


Multiplication is given by concatenation of diagrams, i.e. writing one diagram on top of the other, identifying the bottom row of the upper diagram with the top row of the lower diagram and following the lines from top to bottom or within one row. Parts which have no dot in top or bottom row are replaced by a factor δ . This multiplication is independent of the choice of diagram. We usually omit the labels $1, \dots, r, 1', \dots, r'$.

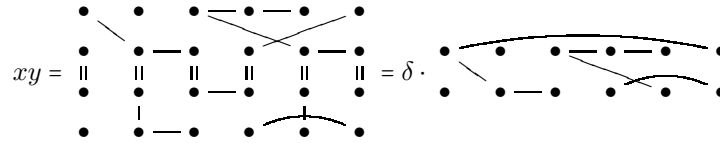
Example. Let



and



Then

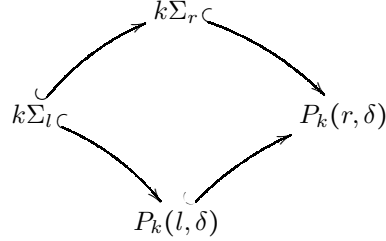


We choose to write all diagrams as follows: First, connect dots of the top row belonging to the same part from left to right. Do the same in the bottom row. Parts which contain both top and bottom row dots will be connected via the respective leftmost dots. Parts connecting top and bottom row are often called *propagating parts* in the literature. The number $\#_p(d)$ of propagating parts of a diagram d is called *propagating number*. We call the actual line connecting a top and a bottom row dot *propagating line*. We denote the top row of a diagram d by $\text{top}(d)$, its bottom row by $\text{bottom}(d)$ and the permutation induced by the propagating

lines by $\Pi(d)$. Note that multiplication of diagrams cannot increase the propagating number, since a propagating part of $x \cdot y$ connects $\text{top}(x)$ to $\text{bottom}(y)$ via $\text{bottom}(x) = \text{top}(y)$, hence $\#_p(x \cdot y) \leq \min\{\#_p(x), \#_p(y)\}$. The unit element of $P_k(r, \delta)$ is given by the set partition $\{\{1, 1'\}, \{2, 2'\}, \dots, \{r, r'\}\} = \begin{array}{cccc} \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \cdots & \vdots \end{array}$.

A diagram consisting of only one row with r dots and arbitrary connections is called *partial diagram*. We have to distinguish certain parts from others; we say they are *labelled* and write the dots as empty circles \circ instead of dots \bullet . We count the labelled parts from left to right, according to the leftmost dot of the part. Let V_n be the vector space with basis all partial diagrams with exactly n labelled parts (and possibly further unlabelled parts). For example, $\bullet \overset{\curvearrowright}{\circ} \bullet - \bullet \quad \circ - \circ \quad \bullet$ is a basis element of V_2 in case $r = 7$; the labelled singleton \circ is the first labelled part, the part $\circ - \circ$ is the second.

2.2. Structural Properties. The group algebra $k\Sigma_r$ is a unitary subalgebra of $P_k(r, \delta)$, where a permutation $\pi \in \Sigma_r$ corresponds to a diagram where all parts are of size 2 and propagating, i.e. each dot of the top row is connected to exactly one dot of the bottom row. For $l < r$, we have different embeddings of $k\Sigma_l$ into $P_k(r, \delta)$



where a smaller partition algebra $P_k(l, \delta)$ is embedded into $P_k(r, \delta)$ by adding dots $l+1, \dots, r$ in the top row and $(l+1)', \dots, r'$ in the bottom row, and attaching the new dots of the top and bottom row, respectively, to the l th, respectively l' th, dot of the diagram in $P_k(l, \delta)$.

Xi showed that $P_k(r, \delta)$ is cellular by considering it as iterated inflation of group algebras of symmetric groups.

Theorem 2 ([Xi, Theorem 4.1]). *The partition algebra $P_k(r, \delta)$ is cellular as an iterated inflation of the form $\bigoplus_{n=0}^r k\Sigma_n \otimes_k V_n \otimes_k V_n$, with respect to the involution i turning a diagram upside down.*

In [HHKP], the partition algebra is one of the main examples for cellularly stratified algebras. For the cellularly stratified structure, we need the existence of idempotents $e_n = 1_{\Sigma_n} \otimes u_n \otimes v_n$ such that $e_n e_m = e_m = e_m e_n$ for $m \leq n$. Let $\delta \neq 0$ and set

$$e_0 := \frac{1}{\delta} \cdot \begin{array}{ccccccc} \bullet^1 & - & \bullet & - & \dots & - & \bullet^r \\ \bullet_{1'} & - & \bullet & - & \dots & - & \bullet_{r'} \end{array} \quad e_n := \begin{array}{ccccccc} \bullet^1 & & \dots & & \bullet & \bullet^n & - & \dots & - & \bullet^r \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ \bullet_{1'} & & & & \bullet & \bullet_{n'} & - & \dots & - & \bullet_{r'} \end{array} \quad \text{for } n \geq 1$$

Theorem 3 ([HHKP, Proposition 2.6]). *The partition algebra $P_k(r, \delta)$ is cellularly stratified with stratification data $(k, V_0, k, V_1, k\Sigma_2, V_2, \dots, k\Sigma_r, V_r)$ and idempotents e_n for all parameters $\delta \in k \setminus \{0\}$.*

Visually speaking, a cellular algebra is cellularly stratified, if there is a chain of two-sided ideals $0 = J_0 \subseteq J_1 \subseteq \dots \subseteq J_r = A$ such that each subquotient J_l/J_{l-1} is an algebra without unit of the form $B_l \otimes V_n \otimes V_n$ for some smaller cellular algebra B_l and vector space V_n . We call these subquotients *layers* of the algebra, and multiplication cannot move to a higher layer.

As a consequence of the cellularly stratified structure, we have that $k\Sigma_r$ is also a quotient of $P_k(r, \delta)$ by the ideal generated by all diagrams with propagating number at most $r - 1$.

From now on, let $A := P_k(r, \delta)$. By abuse of notation, we write $e_n(A/J_{n-1})$ for the $(e_n A e_n, A)$ -bimodule $e_n A / e_n J_{n-1}$, where $J_{n-1} = A e_{n-1} A$ is the two-sided ideal generated by all diagrams d with $\#_p(d) \leq n - 1$. We regard $e_n A$ as left $k\Sigma_n$ -module via the embedding $k\Sigma_n \hookrightarrow e_n A e_n \simeq P_k(n, \delta)$.

The group algebra $k\Sigma_n$ is cellular; we choose as cell modules the dual Specht modules S_λ . Let

$$\mathcal{F}_n(S) = \left\{ M \in k\Sigma_n\text{-mod} \mid \begin{array}{l} M = M_s \supset M_{s-1} \supset \dots \supset M_1 \supset M_0 = 0, \\ M_i/M_{i-1} \simeq S_{\lambda_i} \text{ for some partition } \lambda_i \text{ of } n \end{array} \right\}$$

denote the category of $k\Sigma_n$ -modules admitting a dual Specht filtration.

3. THE BIMODULE $e_l(A/J_{n-1})e_n$

In this section, we study the bimodule $e_l(A/J_{n-1})e_n$, which appears in the restriction of the cell module $(A/J_{n-1})e_n \underset{e_n A e_n}{\otimes} S_\nu$ to $k\Sigma_l\text{-mod}$. By giving multiple isomorphic descriptions for the summands as $(k\Sigma_l, k\Sigma_n)$ -bimodule in this section, we are able to show (in the next section) that the left $k\Sigma_l$ -module $e_l(A/J_{n-1})e_n$ has a dual Specht filtration.

3.1. Notation. Let $n \leq l \leq r$ and let V_n^l be the subspace of V_n generated by all partial diagrams with n labelled parts, where the last $r - l + 1$ dots lie in the same part. We regard $k\Sigma_l$ as the subalgebra of $P_k(r, \delta)$ with basis consisting of all diagrams with top and bottom row consisting of $l - 1$ labelled dots followed by one labelled part of size $r - l + 1$, and l propagating lines connecting the l parts of the top row with the l parts of the bottom row. Let $v, w \in V_n^l$. We say that v is equivalent to w , $v \sim w$, if and only if there is a $\pi \in \Sigma_l$ such that $\pi v = w$, where πv is defined as follows: Write the diagram π on top of v and identify $\text{bottom}(\pi)$ with v . Then πv is the top row of this diagram, where a part is labelled if and only if it contains at least one labelled part. In diagrams, this means that v and w are equivalent, if and only if, for each size, the number of labelled parts and the number of unlabelled parts of v and w coincide, where the last $r - l + 1$ dots count as one.

Example. Let $r = 7, l = 6, n = 2, \pi = (56) \in \Sigma_6$ and

$$v = \circ \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet$$

Then

$$\pi v = \text{top} \left(\begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \circ \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array} \right) = \circ \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet$$

For $v \in V_n^l$, we define d_v to be the diagram with $\text{top}(d_v) = v$, $\text{bottom}(d_v) = \text{bottom}(e_n)$ and $\Pi(d_v) = 1_{k\Sigma_n}$. Let $b \in e_l(A/J_{n-1})e_n$ be a diagram with $\text{top}(b) \sim v$.

By definition, there is a $\pi \in \Sigma_l$ such that $\text{top}(b) = \pi v$. Then $b = \pi d_v \Pi(\pi d_v)^{-1} \Pi(b)$, so we have

- (3.1) Any diagram in $e_l(A/J_{n-1})e_n$ with top row in the equivalence class of v equals $\tau d_v \eta$ for some $\tau \in \Sigma_l, \eta \in \Sigma_n$.

Example. Let $r = 7, l = 6, n = 3, v = \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ$ and

$$b = \begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

Then $\text{top}(b) = (2354)v$ and $\Pi(b) = (132)$. In particular,

$$b = (2354)d_v \Pi((2354)d_v)^{-1} \Pi(b) = \begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

Let U_v be the $(k\Sigma_l, k\Sigma_n)$ -bimodule generated by d_v .

- (3.2) For $w \in V_n^l$ with $w \not\sim v$, we have $U_w \cap U_v = \emptyset$

because a diagram in the intersection would have top row equivalent to v and to w simultaneously. Therefore, every diagram in $e_l(A/J_{n-1})e_n$ lies in exactly one of the U_v 's, see (3.1) and (3.2), and every diagram of U_v is a diagram in $e_l(A/J_{n-1})e_n$ by definition. Hence, the $(k\Sigma_l, k\Sigma_n)$ -bimodule $e_l(A/J_{n-1})e_n$ decomposes into a direct sum $\bigoplus_{v \in V_n^l / \sim} U_v$.

Fix a partial diagram $v \in V_n^l$ and set $d := d_v$. Let α_i be the number of labelled parts of size i and β_i the number of unlabelled parts of size i of v , where the last $r - l + 1$ dots count as one dot. Then $\sum_i (\alpha_i \cdot i) + \sum_i (\beta_i \cdot i) = l$ and $\sum_i \alpha_i = n$. Without loss of generality, assume that the parts of v are ordered as follows: the labelled parts are on the left hand side, the unlabelled parts on the right hand side. The parts are then ordered increasingly from left to right. Let $\mathcal{S}_i^j \subseteq \{1, \dots, l\}$ be the set of dots of v belonging to the j th labelled part of size i and let $\mathcal{T}_i^j \subseteq \{1, \dots, l\}$ be the set of dots of v belonging to the j th unlabelled part of size i . Then $\Pi_\alpha := \prod_{i \geq 1, \alpha_i \neq 0} ((\Sigma_{\mathcal{S}_i^1} \times \dots \times \Sigma_{\mathcal{S}_i^{\alpha_i}}) \rtimes \Sigma_{\alpha_i})$ is the stabilizer of the labelled parts of v and $\Pi_\beta := \prod_{i \geq 1, \beta_i \neq 0} ((\Sigma_{\mathcal{T}_i^1} \times \dots \times \Sigma_{\mathcal{T}_i^{\beta_i}}) \rtimes \Sigma_{\beta_i})$ is the stabilizer of the unlabelled parts of v . In particular, Π_β stabilizes d , while Π_α can rearrange the propagating lines of d . Note that $\Pi_\alpha \simeq \prod_{i \geq 1, \alpha_i \neq 0} (\Sigma_i \wr \Sigma_{\alpha_i})$ and $\Pi_\beta \simeq \prod_{i \geq 1, \beta_i \neq 0} (\Sigma_i \wr \Sigma_{\beta_i})$, where \wr denotes the wreath product.

Example. Let $r = 12, l = 11, n = 3$ and $v = \circ - \circ - \circ - \circ - \circ - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet$. Then $\alpha = (0, 3), \beta = (1, 2)$ and

$$\Pi_\alpha = (\Sigma_{\{1,2\}} \times \Sigma_{\{3,4\}} \times \Sigma_{\{5,6\}}) \rtimes \Sigma_3 \simeq \Sigma_2 \wr \Sigma_3$$

and

$$\Pi_\beta = \Sigma_{\{7\}} \times ((\Sigma_{\{8,9\}} \times \Sigma_{\{10,11\}}) \rtimes \Sigma_2) \simeq \Sigma_1 \times (\Sigma_2 \wr \Sigma_2).$$

3.2. Isomorphic descriptions of the summands. Consider the $(k\Sigma_l, k\Sigma_n)$ -bimodule $k\Sigma_l \otimes_{k\Pi_\alpha \times k\Pi_\beta} k\Sigma_n$, where Π_β acts trivially on $k\Sigma_n$ and the action of Π_α on $k\Sigma_n$ is given by $\zeta \cdot \eta := \Pi(\zeta d)\eta$ for $\zeta \in \Pi_\alpha, \eta \in \Sigma_n$. We have $\text{top}(\zeta d) = \text{top}(d)$, so $\zeta d = d\Pi(\zeta d)$ for $\zeta \in \Pi_\alpha$.

Lemma 1. *The map*

$$\begin{aligned} \psi : k\Sigma_l \otimes_{k\Pi_\alpha \times k\Pi_\beta} k\Sigma_n &\longrightarrow U_v \\ \tau \otimes \eta &\longmapsto \tau d\eta \end{aligned}$$

is an isomorphism of $(k\Sigma_l, k\Sigma_n)$ -bimodules.

Proof. Let $x \in \Pi_\alpha$ and $y \in \Pi_\beta$. Then $yd = d$ and $xd = d\Pi(xd)$. Consider the map $\Psi : k\Sigma_l \times k\Sigma_n \rightarrow U_v$ given by $\Psi(\tau, \eta) = \tau d\eta$. Then $\Psi(\tau xy, \eta) = \tau xy d\eta = \tau d\Pi(xd)\eta = \Psi(\tau, \Pi(xd)\eta) = \Psi(\tau, xy \cdot \eta)$. This shows that ψ is well-defined.

Let $\tau, \tau' \in \Sigma_l$ and $\eta, \eta' \in \Sigma_n$. Then $\psi(\tau'\tau, \eta\eta') = \tau'\tau d\eta\eta' = \tau'\psi(\tau, \eta)\eta'$, so ψ is a bimodule-homomorphism.

The inverse map is given by

$$\begin{aligned} \tilde{\psi} : U_v &\longrightarrow k\Sigma_l \otimes_{k\Pi_\alpha \times k\Pi_\beta} k\Sigma_n \\ b &\longmapsto \tau \otimes \Pi(\tau d)^{-1}\Pi(b) \quad \text{if } \text{top}(b) = \tau v. \end{aligned}$$

We show that $\tilde{\psi}$ is well-defined. If $\text{top}(b) = \tau_1 v = \tau_2 v$, there are $x \in \Pi_\alpha, y \in \Pi_\beta$ such that $\tau_1 = \tau_2 xy$. Then

$$\begin{aligned} \tau_1 \otimes \Pi(\tau_1 d)^{-1}\Pi(b) &= \tau_2 xy \otimes \Pi(\tau_2 xy d)^{-1}\Pi(b) \\ &= \tau_2 \otimes \Pi(xd)\Pi(\tau_2 d\Pi(xd))^{-1}\Pi(b) \\ &= \tau_2 \otimes \Pi(xd)(\Pi(\tau_2 d)\Pi(xd))^{-1}\Pi(b) \\ &= \tau_2 \otimes \Pi(xd)\Pi(xd)^{-1}\Pi(\tau_2 d)^{-1}\Pi(b) \\ &= \tau_2 \otimes \Pi(\tau_2 d)^{-1}\Pi(b). \end{aligned}$$

So, $\tilde{\psi}$ is independent of the choice of τ .

Let $b \in U_v$ be a diagram with $\text{top}(b) = \tau v$ for some $\tau \in \Sigma_l$ and let $\eta \in \Sigma_n$. Then $\psi\tilde{\psi}(b) = \psi(\tau \otimes \Pi(\tau d)^{-1}\Pi(b)) = \tau d\Pi(\tau d)^{-1}\Pi(b) = b$ and $\tilde{\psi}\psi(\tau \otimes \eta) = \tilde{\psi}(\tau d\eta) = \tau \otimes \Pi(\tau d)^{-1}\Pi(\tau d\eta) = \tau \otimes \eta$, so $\tilde{\psi}$ is the inverse of ψ . \square

Set $l_1 := \sum_i \alpha_i \cdot i$ and $l_2 := \sum_i \beta_i \cdot i$, so $l = l_1 + l_2$, $\Pi_\alpha \subset \Sigma_{l_1}$ and $\Sigma_{l_2} \simeq \Sigma_{\bigcup_{i,j} \tau_i^j} \supset \Pi_\beta$. Fix coset representatives $\omega_1, \dots, \omega_t$ of $k\Sigma_l/k\Sigma_{(l_1, l_2)}$. Denote by $X \boxtimes Y \in k\Sigma_{(l_1, l_2)} - \text{mod}$ the exterior tensor product of $X \in k\Sigma_{l_1} - \text{mod}$ and $Y \in k\Sigma_{l_2} - \text{mod}$ given by

$$(\tau_1, \tau_2) \cdot (x \boxtimes y) = \tau_1 x \boxtimes \tau_2 y$$

for $\tau_1 \in \Sigma_{l_1}, \tau_2 \in \Sigma_{l_2}, x \in X, y \in Y$.

Consider the $(k\Sigma_l, k\Sigma_n)$ -bimodule $k\Sigma_l \otimes_{k\Sigma_{(l_1, l_2)}} ((k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \otimes_{k\Pi_\beta} k))$ with right $k\Sigma_n$ -module structure given by

$$(\omega \otimes ((\tau_1 \otimes \eta) \boxtimes (\tau_2 \otimes 1))) \cdot \eta' := \omega \otimes ((\tau_1 \otimes \eta\eta') \boxtimes (\tau_2 \otimes 1))$$

for $\omega \otimes ((\tau_1 \otimes \eta) \boxtimes (\tau_2 \otimes 1)) \in k\Sigma_l \otimes_{k\Sigma_{(l_1, l_2)}} ((k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \otimes_{k\Pi_\beta} k))$ and $\eta' \in \Sigma_n$.

Lemma 2. $k\Sigma_l \otimes_{k\Pi_\alpha \times k\Pi_\beta} k\Sigma_n$ and $k\Sigma_l \otimes_{k\Sigma(l_1, l_2)} ((k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \otimes_{k\Pi_\beta} k))$ are isomorphic as $(k\Sigma_l, k\Sigma_n)$ -bimodules.

Proof. Define

$$\begin{aligned} \Theta : k\Sigma_l \times k\Sigma_n &\longrightarrow k\Sigma_l \otimes_{k\Sigma(l_1, l_2)} ((k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \otimes_{k\Pi_\beta} k)) \\ (\tau, \eta) &\longmapsto \omega_i \otimes ((\tau_1 \otimes \eta) \boxtimes (\tau_2 \otimes 1)) \quad \text{if } \tau = \omega_i \tau_1 \tau_2 \end{aligned}$$

where $\tau_1 \in \Sigma_{l_1}$ and $\tau_2 \in \Sigma_{l_2}$, and let $x \in \Pi_\alpha, y \in \Pi_\beta$. Then $x \in \Sigma_{l_1} \times \{0\} \subset \Sigma_l$, so $\tau_2 x = x \tau_2$ for $\tau_2 \in \Sigma_{l_2}$. Thus, $\Theta(\tau xy, \eta) = \Theta(\omega_i \tau_1 \tau_2 xy, \eta) = \Theta(\omega_i \tau_1 x \tau_2 y, \eta) = \omega_i \otimes ((\tau_1 x \otimes \eta) \boxtimes (\tau_2 y \otimes 1)) = \omega_i \otimes ((\tau_1 \otimes \Pi(xd)\eta) \boxtimes (\tau_2 \otimes 1)) = \Theta(\tau, \Pi(xd)\eta) = \Theta(\tau, xy \cdot \eta)$. Hence, the map

$$\begin{aligned} \theta : k\Sigma_l \otimes_{k\Pi_\alpha \times k\Pi_\beta} k\Sigma_n &\longrightarrow k\Sigma_l \otimes_{k\Sigma(l_1, l_2)} ((k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \otimes_{k\Pi_\beta} k)) \\ \tau \otimes \eta &\longmapsto \omega_i \otimes ((\tau_1 \otimes \eta) \boxtimes (\tau_2 \otimes 1)) \quad \text{if } \tau = \omega_i \tau_1 \tau_2 \end{aligned}$$

is well-defined.

Let $\tau' \in \Sigma_l$ such that $\tau' \omega_i = \omega_j \tau'_1 \tau'_2$ and let $\eta' \in \Sigma_n$. Then

$$\begin{aligned} \theta(\tau' \tau \otimes \eta \eta') &= \theta(\omega_j \tau'_1 \tau'_2 \tau_1 \tau_2 \otimes \eta \eta') \\ &= \theta(\omega_j \tau'_1 \tau_1 \tau'_2 \tau_2 \otimes \eta \eta') \\ &= \omega_j \otimes ((\tau'_1 \tau_1 \otimes \eta \eta') \boxtimes (\tau'_2 \tau_2 \otimes 1)) \\ &= \omega_j \tau'_1 \tau'_2 \otimes ((\tau_1 \otimes \eta \eta') \boxtimes (\tau_2 \otimes 1)) \\ &= \tau' \omega_i \otimes ((\tau_1 \otimes \eta \eta') \boxtimes (\tau_2 \otimes 1)) \\ &= \tau' (\omega_i \otimes ((\tau_1 \otimes \eta) \boxtimes (\tau_2 \otimes 1))) \eta' \\ &= \tau' \theta(\tau \otimes \eta) \eta' \end{aligned}$$

so θ is a homomorphism of $(k\Sigma_l, k\Sigma_n)$ -bimodules.

The inverse is given by

$$\begin{aligned} \theta^{-1} : k\Sigma_l \otimes_{k\Sigma(l_1, l_2)} ((k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \otimes_{k\Pi_\beta} k)) &\longrightarrow k\Sigma_l \otimes_{k\Pi_\alpha \times k\Pi_\beta} k\Sigma_n \\ \tau \otimes ((\vartheta \otimes \eta) \boxtimes (v \otimes 1)) &\longmapsto (\tau \vartheta v \otimes \eta) \end{aligned} \quad \square$$

3.3. The tensor induced module. In this subsection, we find an isomorphic description for $k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n$, which shows that it has a dual Specht filtration as left $k\Sigma_{l_1}$ -module. Let γ be the composition $(1^{\alpha_1}, 2^{\alpha_2}, \dots)$ of l_1 . Then $\Sigma_\gamma = \prod_{i,j} \Sigma_{\mathcal{S}_i^j} \subset \Sigma_{l_1}$ and $\Sigma_\gamma \times \Pi_\beta$ is the stabilizer of d . Fix coset representatives $\sigma_1, \dots, \sigma_s$ of $\Sigma_\alpha \backslash \Sigma_n$ and define a right $k\Sigma_n$ -module structure on $\bigoplus_{i=1}^s (k\Sigma_{l_1} \otimes_{k\Sigma_\gamma} k)$ by

$$(\tau \otimes 1)^{(i)} \cdot \eta := (\tau \hat{\vartheta} \otimes 1)^{(j)} \text{ if } \sigma_i \eta = \vartheta \sigma_j \text{ and } \Pi(\hat{\vartheta} d) = \vartheta$$

for $\tau \in \Sigma_{l_1}, \eta \in \Sigma_n, \vartheta \in \Sigma_\alpha$ and $\hat{\vartheta} \in \Sigma_{l_1}$, where $(\tau \otimes \eta)^{(i)}$ denotes the element $(0, \dots, 0, \tau \otimes \eta, 0, \dots, 0)$ with non-zero entry in the i th position. With this module structure, $\bigoplus_{i=1}^s (k\Sigma_{l_1} \otimes_{k\Sigma_\gamma} k)$ is called *tensor induced module*, cf. [CR, §13].

Lemma 3. *The map*

$$\begin{aligned} \varphi : k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n &\longrightarrow \bigoplus_{i=1}^s (k\Sigma_{l_1} \otimes_{k\Sigma_\gamma} k) \\ \tau \otimes \eta &\longmapsto (\tau \hat{\zeta} \otimes 1)^{(i)} \quad \text{if } \eta = \zeta \sigma_i, \\ &\quad \text{where } \zeta = \Pi(\hat{\zeta}d) \text{ for some } \hat{\zeta} \in \Pi_\alpha \end{aligned}$$

is an isomorphism of $(k\Sigma_{l_1}, k\Sigma_n)$ -bimodules. In particular, $k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n \simeq \bigoplus_{i=1}^s M^\gamma$ as left $k\Sigma_{l_1}$ -modules, so $k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n \in \mathcal{F}_{l_1}(S)$.

Proof. We show that φ is independent of the choice of $\hat{\zeta}$. Let $\hat{\zeta}, \check{\zeta} \in \Pi_\alpha$ such that $\Pi(\hat{\zeta}d) = \Pi(\check{\zeta}d) = \zeta \in \Sigma_\alpha$. Since $\Pi_\alpha \times \Pi_\beta$ is the stabilizer of $\text{top}(d)$, we have $\text{top}(\hat{\zeta}d) = \text{top}(d) = \text{top}(\check{\zeta}d)$, and so $\hat{\zeta}d = \check{\zeta}d$. In particular, there is $(\vartheta, \vartheta') \in \Sigma_\gamma \times \Pi_\beta$, the stabilizer of d , such that $\check{\zeta} = \hat{\zeta}\vartheta\vartheta'$. But $\check{\zeta} = \hat{\zeta}\vartheta\vartheta'$ and $\hat{\zeta}\vartheta$ are elements of Π_α while $\vartheta' \in \Pi_\beta$, so $\vartheta' = 1$. Hence, $(\tau \check{\zeta} \otimes 1)^{(i)} = (\tau \hat{\zeta} \vartheta \otimes 1)^{(i)} = (\tau \hat{\zeta} \otimes 1)^{(i)}$ and φ is independent of the choice of $\hat{\zeta}$.

Let $\Phi : k\Sigma_{l_1} \times k\Sigma_n \rightarrow \bigoplus_{i=1}^s (k\Sigma_{l_1} \otimes_{k\Sigma_\gamma} k)$ with $\Phi(\tau, \eta) = (\tau \hat{\zeta} \otimes 1)^{(i)}$ for $\eta = \zeta \sigma_i$, $\Pi(\hat{\zeta}d) = \zeta$ and let $\xi \in \Pi_\alpha$. Then $\Phi(\tau \xi, \eta) = \Phi(\tau \xi, \zeta \sigma_i) = (\tau \xi \hat{\zeta} \otimes 1)^{(i)}$ and $\Phi(\tau, \Pi(\xi d)\eta) = \Phi(\tau, \Pi(\xi d)\zeta \sigma_i) = (\tau \xi \hat{\zeta} \otimes 1)^{(i)}$, since $\Pi(\xi \hat{\zeta} d) = \Pi(\xi d \Pi(\hat{\zeta}d)) = \Pi(\xi d \zeta) = \Pi(\xi d)\zeta$. So φ is well-defined.

Let $\tau, \tau' \in \Sigma_{l_1}$ and $\eta, \eta' \in \Sigma_n$ such that $\eta = \zeta \sigma_i$ and $\sigma_i \eta' = \zeta' \sigma_j$. Then $\varphi(\tau' \tau \otimes \eta \eta') = \varphi(\tau' \tau \otimes \zeta \zeta' \sigma_j) = (\tau' \tau \hat{\zeta} \otimes 1)^{(j)}$ where $\Pi(\hat{\zeta}d) = \zeta \zeta'$. On the other hand,

$$\begin{aligned} \tau' \varphi(\tau \otimes \eta) \eta' &= \tau' (\tau \hat{\zeta} \otimes 1)^{(i)} \eta' \quad \text{with } \Pi(\hat{\zeta}d) = \zeta \\ &= (\tau' \tau \hat{\zeta} \otimes 1)^{(j)} \quad \text{with } \Pi(\hat{\zeta}d) = \zeta'. \end{aligned}$$

$\Pi(\hat{\zeta} \check{\zeta} d) = \Pi(\hat{\zeta} d \Pi(\check{\zeta} d)) = \Pi(\hat{\zeta} d) \Pi(\check{\zeta} d) = \zeta \zeta' = \Pi(\check{\zeta} d)$, so there is a $\vartheta \in \Sigma_\gamma$ such that $\hat{\zeta} \check{\zeta} = \check{\zeta} \vartheta$. Hence $\tau' \tau \hat{\zeta} \check{\zeta} \otimes 1 = \tau' \tau \check{\zeta} \vartheta \otimes 1 = \tau' \tau \check{\zeta} \otimes 1$ and φ is a homomorphism of $(k\Sigma_{l_1}, k\Sigma_n)$ -bimodules.

The inverse is given by

$$\begin{aligned} \bigoplus_{i=1}^s (k\Sigma_{l_1} \otimes_{k\Sigma_\gamma} k) &\longrightarrow k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n \\ (\tau \otimes 1)^{(i)} &\longmapsto \tau \otimes \sigma_i. \end{aligned}$$

□

3.4. The Foulkes module. We will now study the module $k\Sigma_{l_2} \otimes_{k\Pi_\beta} k$. Let t be the maximal size of an unlabelled part of v and set $\Sigma_{\tilde{\gamma}_i} := \Sigma_{\bigcup_j \mathcal{T}_i^j}$ for $i = 1, \dots, t$ and $\Sigma_{\tilde{\gamma}} := \prod_i \Sigma_{\tilde{\gamma}_i}$. Then $\Pi_\beta \subset \Sigma_{\tilde{\gamma}} \subset \Sigma_{l_2}$.

Lemma 4. *There is an isomorphism*

$$k\Sigma_{l_2} \otimes_{k\Pi_\beta} k \simeq k\Sigma_{l_2} \otimes_{k\Sigma_{\tilde{\gamma}}} (k \boxtimes (k\Sigma_{\tilde{\gamma}_2} \otimes_{k(\Sigma_2 \wr \Sigma_{\beta_2})} k) \boxtimes \dots \boxtimes (k\Sigma_{\tilde{\gamma}_t} \otimes_{k(\Sigma_t \wr \Sigma_{\beta_t})} k))$$

of left $k\Sigma_{l_2}$ -modules.

Proof. Let $\epsilon_1, \dots, \epsilon_u$ be coset representatives of $\Sigma_{l_2}/\Sigma_{\tilde{\gamma}}$ and $\tau = \epsilon_i \tau_2 \dots \tau_t$ with $\tau_j \in \Sigma_{\tilde{\gamma}_j}$. Then the assignment

$$\tau \otimes 1 \mapsto \epsilon_i \otimes (1 \boxtimes (\tau_2 \otimes 1) \boxtimes \dots \boxtimes (\tau_t \otimes 1))$$

defines the isomorphism, like in Lemma 2. \square

The module $H^{(a^m)} := k\Sigma_{am} \otimes_{k(\Sigma_a \wr \Sigma_m)} k$ is called *Foulkes module*. If the characteristic of the field k is strictly greater than m , or zero, the Foulkes module is isomorphic to a direct summand of the permutation module $M^{(a^m)}$, as mentioned in [Gia]. We will give a proof of this statement in Lemma 5. In smaller positive characteristic, this is not true. In general, it is not known whether or not a Foulkes module $H^{(a^m)}$ has a Specht filtration in the case $0 < \text{char } k \leq m$ and $a > 3$. The case $a = 2$ was solved in [Pag] for arbitrary characteristic of the field.

Lemma 5. *If $\text{char } k = 0$ or $\text{char } k > m$, the Foulkes module*

$$H^{(a^m)} = k\Sigma_{am} \otimes_{k(\Sigma_a \wr \Sigma_m)} k$$

is isomorphic to a direct summand of the permutation module $M^{(a^m)}$.

Proof. We came across the wreath product $\Sigma_a \wr \Sigma_m$ as the stabilizer of the m (unlabelled) parts of size a of a partial diagram. The Foulkes module $k\Sigma_{am} \otimes_{k(\Sigma_a \wr \Sigma_m)} k$ has a vector space basis indexed by left cosets $\Sigma_{am}/(\Sigma_a \wr \Sigma_m)$. Such a coset decides which dots belong to the same part. Thus, $k\Sigma_{am} \otimes_{k(\Sigma_a \wr \Sigma_m)} k$ has a vector space basis of set partitions of the form

$$\{\{x_1, \dots, x_a\}, \dots, \{x_{(m-1)a+1}, \dots, x_{ma}\}\}$$

with $x_i \in \{1, \dots, am\}, x_i \neq x_j$ for $i \neq j$.

Recall that the permutation module $M^{(a^m)}$ has a basis of (a^m) -tabloids. Define maps $M^{(a^m)} \xrightleftharpoons[\Psi]{\Phi} H^{(a^m)}$ with

$$\begin{array}{c} \begin{array}{|c|c|c|} \hline x_1 & \dots & x_a \\ \hline & \vdots & \\ \hline x_{(m-1)a+1} & \dots & x_{ma} \\ \hline \end{array} & \xrightarrow{\Phi} & \{\{x_1, \dots, x_a\}, \dots, \{x_{(m-1)a+1}, \dots, x_{ma}\}\} \\ \\ \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \sigma * \begin{array}{|c|c|c|} \hline x_1 & \dots & x_a \\ \hline & \vdots & \\ \hline x_{(m-1)a+1} & \dots & x_{ma} \\ \hline \end{array} & \xleftarrow{\Psi} & \{\{x_1, \dots, x_a\}, \dots, \{x_{(m-1)a+1}, \dots, x_{ma}\}\} \end{array}$$

where the $*$ -action of Σ_m permutes the rows of a tabloid. The $*$ and \cdot actions commute: Let $\sigma \in \Sigma_m, \tau \in \Sigma_{am}$ and x_i in row k of the tabloid x . If $\sigma(k) = l$, then x_i is in row l of $\sigma * x$, so $x_{\tau(i)}$ is in row l of $\tau \cdot (\sigma * x)$. On the other hand, $x_{\tau(i)}$ is in row k of $\tau \cdot x$ and therefore in row l of $\sigma * (\tau \cdot x)$.

We have

$$\begin{aligned}
 & \Phi \left(\tau \cdot \begin{array}{c|c|c} x_1 & \cdots & x_a \\ \hline & & \vdots \\ \hline x_{(m-1)a+1} & \cdots & x_{ma} \end{array} \right) \\
 &= \Phi \left(\begin{array}{c|c|c} x_{\tau(1)} & \cdots & x_{\tau(a)} \\ \hline & & \vdots \\ \hline x_{\tau((m-1)a+1)} & \cdots & x_{\tau(ma)} \end{array} \right) \\
 &= \{ \{x_{\tau(1)}, \dots, x_{\tau(a)}\}, \dots, \{x_{\tau((m-1)a+1)}, \dots, x_{\tau(ma)}\} \} \\
 &= \tau \cdot \{ \{x_1, \dots, x_a\}, \dots, \{x_{(m-1)a+1}, \dots, x_{ma}\} \} \\
 &= \tau \Phi \left(\begin{array}{c|c|c} x_1 & \cdots & x_a \\ \hline & & \vdots \\ \hline x_{(m-1)a+1} & \cdots & x_{ma} \end{array} \right)
 \end{aligned}$$

and

$$\begin{aligned}
 & \Psi \left(\tau \cdot \{ \{x_1, \dots, x_a\}, \dots, \{x_{(m-1)a+1}, \dots, x_{ma}\} \} \right) \\
 &= \Psi \left(\{ \{x_{\tau(1)}, \dots, x_{\tau(a)}\}, \dots, \{x_{\tau((m-1)a+1)}, \dots, x_{\tau(ma)}\} \} \right) \\
 &= \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \sigma * \begin{array}{c|c|c} x_{\tau(1)} & \cdots & x_{\tau(a)} \\ \hline & & \vdots \\ \hline x_{\tau((m-1)a+1)} & \cdots & x_{\tau(ma)} \end{array} \\
 &= \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \sigma * \left(\tau \cdot \begin{array}{c|c|c} x_1 & \cdots & x_a \\ \hline & & \vdots \\ \hline x_{(m-1)a+1} & \cdots & x_{ma} \end{array} \right) \\
 &= \tau \cdot \left(\frac{1}{m!} \sum_{\sigma \in \Sigma_m} \sigma * \begin{array}{c|c|c} x_1 & \cdots & x_a \\ \hline & & \vdots \\ \hline x_{(m-1)a+1} & \cdots & x_{ma} \end{array} \right) \\
 &= \tau \cdot \Psi \left(\{ \{x_1, \dots, x_a\}, \dots, \{x_{(m-1)a+1}, \dots, x_{ma}\} \} \right).
 \end{aligned}$$

Hence, Ψ and Φ are $k\Sigma_{am}$ -module homomorphisms. Φ is surjective and $\Phi\Psi$ is the identity on $H^{(a^m)}$, so Φ is a split epimorphism. \square

As a direct corollary, we get

Corollary 6. *If $\text{char } k = 0$ or $\text{char } k > m$, the indecomposable direct summands of the Foulkes module $H^{(a^m)} = k\Sigma_{am} \otimes_{k(\Sigma_a \wr \Sigma_m)} k$ are Young modules. In particular, $H^{(a^m)}$ is both Specht and dual Specht filtered.*

Corollary 7. *If $\text{char } k = 0$ or $\text{char } k > \max \beta_i$, then $k\Sigma_{l_2} \otimes_{k\Pi_\beta} k \in \mathcal{F}_{l_2}(S)$.*

Proof. By Lemma 4, $k\Sigma_{l_2} \otimes_{k\Pi_\beta} k$ is induced from an exterior tensor product of Foulkes modules. Corollary 6 shows that the Foulkes modules are dual Specht filtered, provided the characteristic of the field is large enough. The characteristic-free version of the Littlewood-Richardson rule [JP] then says that the exterior tensor product of Foulkes modules has a dual Specht filtration. \square

4. RESTRICTION OF CELL MODULES

We are now able to put the results about tensor induced and Foulkes modules together to show that the restriction of a cell module of $P_k(r, \delta)$ to a group algebra of a symmetric group with index $l \leq r$ is dual Specht filtered.

The factor $k\Sigma_{2\beta_2} \otimes_{k(\Sigma_2 \wr \Sigma_{\beta_2})} k$ is the stabilizer of unlabelled parts of size 2 and it is dual Specht filtered by [Pag, Proposition 8]. For $i > 2$, the factor $k\Sigma_{i\beta_i} \otimes_{k(\Sigma_i \wr \Sigma_{\beta_i})} k$ is dual Specht filtered if $\text{char} k = 0$ or $\text{char} k > \beta_i$ by Corollary 6. The maximal amount of unlabelled parts of a certain size greater than two occurs in the summands U_v , where v consists of n labelled singletons and $\lfloor \frac{r-n}{3} \rfloor$ unlabelled parts of size 3. The remaining 0, 1 or 2 dots form additional unlabelled parts.

Proposition 8. *Let $\text{char} k = 0$ or $\text{char} k > \lfloor \frac{r-n}{3} \rfloor$. Then $e_l(A/J_{n-1})e_n \in \mathcal{F}_l(S)$.*

Proof. By Lemmas 1 and 2, the summands of $e_l(A/J_{n-1})e_n$ are of the form

$$k\Sigma_l \otimes_{k\Sigma(l_1, l_2)} ((k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \otimes_{k\Pi_\beta} k)),$$

where $\Pi_\alpha \simeq k \prod_i (\Sigma_i \wr \Sigma_{\alpha_i})$ and $\Pi_\beta \simeq k \prod_i (\Sigma_i \wr \Sigma_{\beta_i})$, $l_1 = \sum_i \alpha_i \cdot i$ and $l_2 = \sum_i \beta_i \cdot i$. Lemma 3 shows that $k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n \in \mathcal{F}_{l_1}(S)$ and by Corollary 7, we have that $k\Sigma_{l_2} \otimes_{k\Pi_\beta} k \in \mathcal{F}_{l_2}(S)$ in case $\text{char} k > \max \beta_i$ or zero. Since we are looking at the whole bimodule $e_l(A/J_{n-1})e_n$ and not just its summands U_v , we have to consider all possible top rows. By the above arguments, we have that the maximal amount of unlabelled parts of size ≥ 3 is $\lfloor \frac{r-n}{3} \rfloor$. Hence, we have to assume $\text{char} k > \lfloor \frac{r-n}{3} \rfloor$. The characteristic-free version of the Littlewood-Richardson rule [JP] then says that $k\Sigma_l \otimes_{k\Sigma(l_1, l_2)} ((k\Sigma_{l_1} \otimes_{k\Pi_\alpha} k\Sigma_n) \boxtimes (k\Sigma_{l_2} \otimes_{k\Pi_\beta} k)) \in \mathcal{F}_l(S)$. \square

Theorem (Theorem 1). *Let A be the partition algebra $P_k(r, \delta)$ and $n \leq l \leq r$. Let $\text{char} k = 0$ or $\text{char} k > \lfloor \frac{r-n}{3} \rfloor$ and $X \in k\Sigma_n - \text{mod}$. Then the $k\Sigma_l$ -module $e_l(A/J_{n-1})e_n \otimes_{e_n A e_n} X$ is in $\mathcal{F}_l(S)$. In particular, restrictions of cell modules to $k\Sigma_l - \text{mod}$ are dual Specht filtered.*

Proof. By Lemmas 1, 2, 3 and 4, $e_l(A/J_{n-1})e_n$ decomposes as $(k\Sigma_l, k\Sigma_n)$ -bimodule into a direct sum of modules of the form

$$k\Sigma_l \otimes_{k\Sigma_{l_1} \times k\Sigma_{l_2}} ((\bigoplus_{i=1}^s (k\Sigma_{l_1} \otimes_{k\Sigma_{\gamma_i}} k)) \boxtimes (k\Sigma_{l_2} \otimes_{k\Sigma_{\tilde{\gamma}_i}} (k \boxtimes (k\Sigma_{\tilde{\gamma}_2} \otimes_{k(\Sigma_2 \wr \Sigma_{\beta_2})} k) \boxtimes \dots \boxtimes (k\Sigma_{\tilde{\gamma}_t} \otimes_{k(\Sigma_t \wr \Sigma_{\beta_t})} k)))).$$

Hence, an element of $e_l(A/J_{n-1})e_n \otimes_{e_n A e_n} X \simeq e_l(A/J_{n-1})e_n \otimes_{k\Sigma_n} X$ is of the form

$$\begin{aligned} & \omega \otimes \left(\sum_{i=1}^s (\pi_i \otimes 1)^{(i)} \boxtimes (v \otimes (1 \boxtimes (\tau_2 \otimes 1) \boxtimes \dots \boxtimes (\tau_t \otimes 1))) \right) \otimes x \\ &= \omega \otimes \left(\sum_{i=1}^s (\pi_i \otimes 1)^{(1)} \sigma_i \boxtimes (v \otimes (1 \boxtimes (\tau_2 \otimes 1) \boxtimes \dots \boxtimes (\tau_t \otimes 1))) \right) \otimes x \\ &= \omega \otimes \left(\sum_{i=1}^s (\pi_i \otimes 1)^{(1)} \boxtimes (v \otimes (1 \boxtimes (\tau_2 \otimes 1) \boxtimes \dots \boxtimes (\tau_t \otimes 1))) \right) \otimes \sigma_i x \\ &= \omega \otimes \left(\left(\sum_{i=1}^s \pi_i \otimes 1 \right)^{(1)} \boxtimes (v \otimes (1 \boxtimes (\tau_2 \otimes 1) \boxtimes \dots \boxtimes (\tau_t \otimes 1))) \right) \otimes \sigma_i x \end{aligned}$$

with $\omega \in k\Sigma_l$, $\pi_i \in k\Sigma_{l_1}$, $v \in k\Sigma_{l_2}$ and $\tau_i \in k\Sigma_{\tilde{\gamma}_i}$. Hence the summands of $e_l(A/J_{n-1})e_n \otimes_{e_n A e_n} X$ are isomorphic to

$$Y := Z \otimes_k (k \otimes_{k\Sigma_\alpha} k\Sigma_n \otimes_{k\Sigma_n} X),$$

where Z is the module $k\Sigma_l \otimes_{k\Sigma_{(l_1, l_2)}} ((k\Sigma_{l_1} \otimes k) \boxtimes (k\Sigma_{l_2} \otimes k))$. The factor $k\Sigma_{l_1} \otimes k$ is clearly dual $k\Sigma_{l_1}$ -Specht filtered, as it is a permutation module for $k\Sigma_{l_1}$. The factor $k\Sigma_{l_2} \otimes k$ is dual $k\Sigma_{l_2}$ -Specht filtered by Corollary 7 if $\text{char } k > \max \beta_i$ or $\text{char } k = 0$. Now we can apply the characteristic-free version of the Littlewood-Richardson rule again to have $Z \in \mathcal{F}_l(S)$. Then the left $k\Sigma_l$ -module $Y = \bigoplus_{i=1}^h Z$ is in $\mathcal{F}_l(S)$, where $h = \dim(k \otimes_{k\Sigma_\alpha} X)$. \square

ACKNOWLEDGEMENTS

The author is grateful to Steffen Koenig for useful discussions and proof reading of this article, and to Eugenio Giannelli for discussions on Foulkes modules. This article is based on the author's PhD thesis, which was partly funded by DFG-SPP 1489.

REFERENCES

- [CR] C. W. Curtis and I. Reiner. *Methods of representation theory. Vol. I*. John Wiley & Sons, Inc., New York, 1981. With applications to finite groups and orders, Pure and Applied Mathematics, A Wiley-Interscience Publication.
- [Gia] E. Giannelli. On permutation modules and decomposition numbers of the symmetric group. *J. Algebra*, 422:427–449, 2015.
- [HR] T. Halverson and A. Ram. Partition algebras. *European J. Combin.*, 26(6):869–921, 2005.
- [HHKP] R. Hartmann, A. Henke, S. König, and R. Paget. Cohomological stratification of diagram algebras. *Math. Ann.*, 347(4):765–804, 2010.
- [HP] R. Hartmann and R. Paget. Young modules and filtration multiplicities for brauer algebras. *Math. Z.*, 254(2):333–357, 2006.
- [JP] G. D. James and M. H. Peel. Specht series for skew representations of symmetric groups. *J. Algebra*, 56(2):343–364, 1979.
- [Jon] V. F. R. Jones. The Potts model and the symmetric group. In *Subfactors (Kyuzeso, 1993)*, pages 259–267. World Sci. Publ., River Edge, NJ, 1994.
- [KX] S. König and C.C. Xi. When is a cellular algebra quasi-hereditary? *Math. Ann.*, 315(2):281–293, 1999.
- [Mar] P. Martin. Temperley-Lieb algebras for nonplanar statistical mechanics—the partition algebra construction. *J. Knot Theory Ramifications*, 3(1):51–82, 1994.
- [Pag] R. Paget. A family of modules with Specht and dual Specht filtrations. *J. Algebra*, 312(2):880–890, 2007.
- [Pau] I. Paul. *Structure theory for cellularly stratified diagram algebras*. PhD thesis, Stuttgart, 2015. <http://dx.doi.org/10.18419/opus-5153>.
- [Xi] C.C. Xi. Partition algebras are cellular. *Compositio Math.*, 119(1):99–109, 1999.